# ДИФРАКЦИОННАЯ ОПТИКА, ОПТИЧЕСКИЕ ТЕХНОЛОГИИ

## POLARIZATION PROPERTIES OF THREE-DIMENSIONAL ELECTROMAGNETIC GAUSSIAN SCHELL-MODEL SOURCES

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***Abstract***

The polarization properties of the recently introduced three-dimensional electromagnetic Gaussian Schell-model sources [Opt. Lett. 42, 1792 (2017)] are examined. Both cases of uniform and non-uniform polarization are considered. The three-dimensional polarization states are characterized via the eigenvalues of a 3×3 source polarization matrix and, more specifically, via the indices of polarimetric purity. We show that the considered sources exhibit a variety of polarization states throughout their volumes conveniently controlled by several physically accessible source parameters.

*Keywords:* polarization, coherence.

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### *Introduction*

The class of wide-sense statistically stationary, three-dimensional, electromagnetic Gaussian Schell-model [3D EM GSM] sources has been recently introduced theoretically [1] and simulated by the Monte-Carlo method [2]. To our knowledge, these sources are the only model random sources, besides the blackbody radiation [3 – 6] (see also review article [7]) in which all three-components of the electric field exhibit random oscillations. The preliminary analysis of the spectral density distribution of the 3D EM GSM sources has been given in Ref. [1] but its polarization content remains unknown.

Unlike the well-understood 2D polarization characterization [8], the complete polarimetric analysis of the 3D random fields still remains obscure. However, it has recently become apparent that certain quantities such as the eigenvalues of the 3D polarization matrix [9, 10] and based on them indices of polarimetric purity [11] provide comprehensive polarization information and reduce to conventional definitions and limiting states known from the classic 2D theory (theory of random EM beams) [8].

The aim of this paper is to elucidate the possible 3D polarization states of the 3D EM GSM sources and relate them to the source parameters. First, we state the conditions on source parameters that ensure the uniform polarization in the source volume. Further, for the situations when the 3D polarization state is uniform in the source volume we derive conditions on the source parameters for the two limiting cases of complete and zero polarization. We also specify the set of parameters for which the 3D sources reduce to the corresponding 2D sources, i.e., the sources radiating the EM beam-like fields. In cases when the polarization state is not uniform within the source volume we offer the set of numerical examples covering typical variation of indices of polarimetric purity with radial position.

### *1. Description of the 3D random optical fields*

The second-order coherence and polarization properties of the wide-sense statistically stationary 3D EM sources and fields they radiate can be characterized with the help of the 3×3 cross-spectral density (CSD) matrix [12]. The elements of the CSD matrix within the source volume at positions $\mathbf{r}_1$ and $\mathbf{r}_2$ and angular frequency ω have form

$$\vec{\vec{W}}(\mathbf{r}_1,\mathbf{r}_2;\omega)=\left\langle \vec{E}^{\dagger}(\mathbf{r}_1;\omega)\vec{E}(\mathbf{r}_2;\omega)\right\rangle_{\omega}, \quad (1)$$

where dagger stands for Hermitian adjoint, angular brackets with subscript ω denote average taken over the ensemble of monochromatic realizations and

$$\vec{E}(\mathbf{r};\omega)=[E_x(\mathbf{r};\omega),E_y(\mathbf{r};\omega),E_z(\mathbf{r};\omega)] \quad (2)$$

is the fluctuating 3D electric vector-field (see Fig. 1).

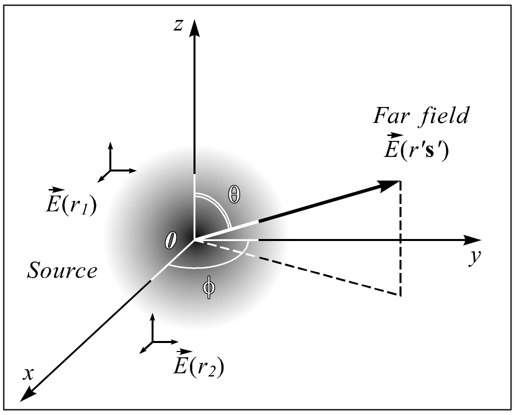


*Fig. 1. Illustrating notations relating to the 3D EM sources and fields radiating from them*

For brevity, the dependence on angular frequency will be omitted throughout the text. The characteristic equation

$$\det[\vec{\vec{W}}(\mathbf{r},\mathbf{r})-\lambda(\mathbf{r})\vec{\vec{I}}_3]=0\ , \quad (3)$$

where the CSD matrix is calculated at coinciding arguments $\mathbf{r}_1=\mathbf{r}_2=\mathbf{r}$ and $\vec{I}_3$ is the 3D identity matrix, leads to three, generally position dependent, eigenvalues $\lambda_1(\mathbf{r})$, $\lambda_2(\mathbf{r})$ and $\lambda_3(\mathbf{r})$ all being real and non-negative due to the Hermitian nature of the CSD matrix at a single argument.

The explicit expressions for the eigenvalues based on the Cardano's formulas in terms of the CSD matrix elements can be found in Ref. [13]. Without loss of generality, we assume here that $\lambda_1(\mathbf{r})\geq\lambda_2(\mathbf{r})\geq\lambda_3(\mathbf{r})$. The two indices of polarimetric purity: the degree of polarization $P_1$ and the degree of directionality $P_2$ are given by the following expressions [11] (see also [13–15]):

$$P_1(\mathbf{r})=\frac{\lambda_1(\mathbf{r})-\lambda_2(\mathbf{r})}{\lambda_1(\mathbf{r})+\lambda_2(\mathbf{r})+\lambda_3(\mathbf{r})}, \tag{4}$$

and

$$P_2(\mathbf{r})=\frac{\lambda_1(\mathbf{r})+\lambda_2(\mathbf{r})-2\lambda_3(\mathbf{r})}{\lambda_1(\mathbf{r})+\lambda_2(\mathbf{r})+\lambda_3(\mathbf{r})}, \tag{5}$$

$0\leq P_1\leq P_2\leq 1$, where the trace of the CSD matrix

$$\begin{aligned}S(\mathbf{r})&=Tr[\overleftrightarrow{W}(\mathbf{r},\mathbf{r})]=\\&=\lambda_1(\mathbf{r})+\lambda_2(\mathbf{r})+\lambda_3(\mathbf{r}),\end{aligned} \tag{6}$$

represents the spectral density (average intensity) at position $\mathbf{r}$. The first index, $P_1$, describes the fraction of power in the purely polarized state. This parameter reduces to the usual degree of polarization for beam-like fields [16]. The second index, $P_2$, gives the fraction of power not included in the unpolarized component, i.e. it contains power of the purely polarized and the 2D unpolarized components. Index $P_2$ may be considered as a directionality or stability degree of the plane of polarization (determined in Ref. [9]). In the 3D field case the state of polarization is still elliptical (in general) but instead of the well-defined plane of the ellipse, as in 2D case, there is an average (preferential) plane of polarization. In 3D case the orientation of this plane fluctuates in time and space, and according to Ref. [11] $P_2$ parameter gives the degree of stability of this plane at a given point. Parameter $P_2$ reduces to unity (perfect stability) in the 2D case (see Ref. [11] for the thorough interpretation of both indices).

## *2. Three-dimensional Electromagnetic Gaussian Schell-model sources*

The elements of the CSD matrix of the 3D EM GSM sources have form:

$$W_{ij}(\mathbf{r}_1,\mathbf{r}_2)=\sqrt{S_i(\mathbf{r}_1)}\sqrt{S_j(\mathbf{r}_2)}\mu_{ij}(\mathbf{r}_2-\mathbf{r}_1), \tag{7}$$

where $(i,j=x,y,z)$ and the components of the spectral density and their correlation coefficients are defined as

$$S_i(\mathbf{r})=A_i^2\exp\left[-\frac{\mathrm{r}^2}{2\sigma_\mathrm{i}^2}\right], \tag{8}$$

$$\mu_{ij}(\mathbf{r}_1,\mathbf{r}_2)=B_{ij}\exp\left[-\frac{(\mathbf{r}_2-\mathbf{r}_1)^2}{2\delta_\mathrm{ij}^2}\right], \tag{9}$$

where spectral amplitudes $A_i$, correlation factors $B_{ij}$ between $i$ and $j$ field components, the r.m.s. widths of spectral density components $\sigma_i$ and the r.m.s. widths $\delta_{ij}$ of the degrees of correlation between $i$ and $j$ field components are, possibly, frequency-dependent parameters. The generally complex coefficients $B_{ij}=b_{ij}\exp[i\phi_{ij}]$ must obey constraints [1], [2]:

$$\begin{aligned}&|b_{ij}|\leq 1,\quad b_{ii}=1,\\&\varphi_{ii}=0,\quad \varphi_{ij}=-\varphi_{ji},\quad \varphi_{ij}+\varphi_{jk}=-\varphi_{ki},\end{aligned} \tag{10}$$

where the last cyclic property of the phase does not have the analog in the 2D case. Further, parameters $\delta_{ij}$ and $b_{ij}$ must also satisfy the realizability condition [1]

$$\max\{\delta_{ii},\delta_{jj}\}\leq\delta_{ij}\leq\min\{\delta_{ii},\delta_{jj}\}(2b_{ij})^{-1/3}, \tag{11}$$

but the r.m.s. widths $\delta_{ij}$ have no effect on the source polarization properties. However, they would affect the polarimetric content of radiation emitted from these sources.

## *3. Uniformly polarized 3D EM GSM sources*

From Eqs. (4), (5) and (7) – (9) we readily see that it suffices for a 3D EM GSM source to have the uniform polarization state (in terms of $P_1$ and $P_2$) everywhere in its volume if the r.m.s. widths of the spectral density components are equal, i.e.,

$$\sigma_x=\sigma_y=\sigma_z=\sigma. \tag{12}$$

Indeed, in this case, the spatial distributions of all elements of the CSD tensor (at a single point) are the same:

$$W_{ij}(\mathbf{r},\mathbf{r})=A_iA_jB_{ij}\exp\left[-\frac{\mathrm{r}^2}{2\sigma^2}\right], \tag{13}$$

where $r=|\mathbf{r}|$ and the eigenvalues of the CSD matrix with elements (13), with and without exponential term, are equal. Hence the characteristic equation (3) reduces to

$$\lambda^3+b\lambda^2+c\lambda+d=0, \tag{14}$$

with

$$\begin{aligned}b&=-(A_x^2+A_y^2+A_z^2),\\c&=A_x^2A_y^2(1-b_{xy}^2)+A_y^2A_z^2(1-b_{yz}^2)+\\&+A_x^2A_z^2(1-b_{xz}^2),\\d&=A_x^2A_y^2A_z^2(b_{xy}^2+b_{yz}^2+b_{xz}^2-1-2b_{xy}^2b_{yz}^2b_{xz}^2),\end{aligned} \tag{15}$$

taking on constant values in the source volume. In derivation of Eq. (15) relations (10) between phases $\phi_{ij}$ of the correlation coefficients $B_{ij}$ have been employed making eigenvalues independent of any of them. At this point we conclude that if polarization of the 3D EM GSM source is uniform in its volume (in terms of the indices of polarimetric purity, $P_1$ and $P_2$) then it is entirely described by 6 real-valued parameters: $A_x$, $A_y$, $A_z$ and $b_{xy}$, $b_{yz}$ and $b_{xz}$. Further, according to the Cardano's formulas for cubic equation, we get the following expressions for the three eigenvalues [17], with $n = 0, 1, 2$:

$$\lambda_n = 2\sqrt{-\frac{p}{3}}\cos\left[\frac{1}{3}\arccos\left(\frac{3q}{2p}\sqrt{-\frac{3}{p}} - \frac{2\pi n}{3}\right)\right], \quad (16)$$

$$\begin{aligned} p &= \frac{3c-b^2}{3} = \\ &= A_x^2A_y^2\left(1-b_{xy}^2\right) + A_y^2A_z^2\left(1-b_{yz}^2\right) + \\ &+ A_x^2A_z^2\left(1-b_{xz}^2\right) - \frac{1}{3}\left(A_x^2+A_y^2+A_z^2\right)^2, \end{aligned} \quad (17)$$

$$\begin{aligned} q &= \frac{2b^3-9bc+27d}{27} = \\ &= \frac{1}{27}\Big[-\left(A_x^2+A_y^2+A_z^2\right)^3 + 9\left(A_x^2+A_y^2+A_z^2\right)\times \\ &\times\left\{A_x^2A_y^2\left(1-b_{xy}^2\right)+A_y^2A_z^2\left(1-b_{yz}^2\right)+A_x^2A_z^2\left(1-b_{xz}^2\right)\right\}\Big] + \\ &+ A_x^2A_y^2A_z^2\left(b_{xy}^2+b_{yz}^2+b_{xz}^2-1-2b_{xy}^2b_{yz}^2b_{xz}^2\right). \end{aligned} \quad (18)$$

Let us now list several important cases of uniform polarization and the corresponding values of the eigenvalues and the indices.

(1) For the case when $A_i = 0$ for two out of three indices, say, $y$ and $z$, and $A_x > 0$ ($A_x = 1$ with power normalization), $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 0$ and $P_1 = P_2 = 1$ leading to the case of complete polarization (in the 3D sense).

(2) If $A_x = A_y = A_z = 1$ and $b_{xy} = b_{yz} = b_{xz} = 1$, then $\lambda_1 = 3$, $\lambda_2 = \lambda_3 = 0$ and $P_1 = P_2 = 1$ also giving complete polarization.

(3) If $A_x = A_y = A_z = 1$, but $b_{xy} = b_{yz} = b_{xz} = 0$, $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and $P_1 = P_2 = 0$, corresponding to the case of unpolarized light (in the 3D sense).

Two other important cases arise when two out of three amplitudes are nonzero and are the same, e.g.

(4) $A_x = A_y = 1$, $A_z = 0$, and $b_{xy} = b_{yz} = b_{xz} = 0$. In this case $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = 0$, $P_1 = 0$ but $P_2 = 1$, and, thus, the source reduces to uniformly unpolarized 2D EM GSM beam [15].

(5) If $A_x = A_y = 1$, $A_z = 0$, but instead, $b_{xy} = 1$, then $\lambda_1 = 2$, $\lambda_2 = \lambda_3 = 0$ and $P_1 = P_2 = 1$, being the uniformly polarized 2D EMGSM beam [18].

We also mention an important case of partial polarization.

(6) In the situations when $A_x = A_y = A_z = 1$ but $b_{xy} = b_{yz} = b_{xz} = b_c$ the polarization properties entirely depend on the value of $b_c$. Indeed in this case: $\lambda_1 = 1 + 2b_c$, $\lambda_{2,3} = 1 - b_c$, and, hence, $P_1 = P_2 = b_c$. This result is independent of the relative phases $\phi_{ij}$ of the electric field components as long as the relations (10) are met. Further, according to the realizability conditions (11), as $b_c$ increases to unity, relation $\delta_{ij} = \delta_{ij} = \delta_{ij}$ for $(i, j = x, y, z)$ must hold leading to fully polarized source with the same correlations in $x$, $y$ and $z$ directions.

### 4. Examples of non-uniformly polarized 3D EM GSM sources

We will now illustrate a possible non-uniform distribution of the spectral density and the polarization indices of the 3D EM GSM sources by numerical examples. Unless other values are specified in figure caption, the selected parameters' values of the source are as follows:

$$\begin{aligned} &A_x = 1, \quad A_y = 1.5, \quad A_z = 2, \\ &\sigma_x = 4cm, \quad \sigma_y = 5cm, \quad \sigma_z = 10cm, \\ &b_{xy} = 0.222, \quad b_{xz} = 0.286, \quad b_{xz} = 0.539, \\ &\varphi_{xy} = \pi/2, \quad \varphi_{xz} = \pi/3, \quad \varphi_{yz} = -\pi/6. \end{aligned} \quad (19)$$

This set of values satisfies conditions (10) and (11) (see also Ref. [2]). The rest of the parameters are not given since they do not enter calculations relating either to the spectral density or the polarization state.

Fig. 2 shows the typical 3D distributions of the spectral density (solid blue curves) together with the two indices of polarimetric purity, $P_1$ and $P_2$, in the source volume (dashed purple and dotted green curves, respectively). While the spectral density has a profile close to Gaussian (since the r.m.s. spreads are not too different from each other, see [1] for other cases), the polarimetric indices can experience drastic variation with radial distance r. The degree of directionality $P_2$ always takes on values larger than the degree of polarization $P_1$ and exhibits less variation. At the edge of the source both indices approach unity. We also note that according to case (3) of uniformly polarized sources ($A_x = A_y = A_z = 1$, but $b_{xy} = b_{yz} = b_{xz} = 0$) both indices must vanish but as it is clear from Fig. 2*a* it is not generally the case for the non-uniformly polarized sources: both indices vanish only in the center of the source.

### Summary

We have examined the polarization properties of the 3D EM GSM sources in terms of the two indices of polarimetric purity: the degree of polarization $P_1$ (in the 3D sense) and the degree of directionality $P_2$, both being the invariants of the CSD matrix at a single spatial argument. In general, 17 real-valued, independent parameters ($A_x$, $A_y$, $A_z$, $b_{xy}$, $b_{yz}$, $b_{xz}$, $\phi_{xy}$, $\phi_{yz}$, $\delta_{xx}$, $\delta_{yy}$, $\delta_{zz}$, $\delta_{xy}$, $\delta_{yz}$, $\delta_{xz}$, $\sigma_x$, $\sigma_y$, $\sigma_z$) constrained to some extent by the realizability conditions are shown to determine the spectral, correlation and polarimetric properties of the source. Only 9 out of 17 parameters ($A_x$, $A_y$, $A_z$, $b_{xy}$, $b_{yz}$, $b_{xz}$, $\sigma_x$, $\sigma_y$, $\sigma_z$) affect the polarimetric indices, and this number reduces to 6 ($A_x$, $A_y$, $A_z$, $b_{xy}$, $b_{yz}$, $b_{xz}$) in case of uniform polarization. The two independent phases of correlation coefficients, say $\phi_{xy}$ and $\phi_{yz}$, do not affect the indices but can affect other (not rotationally invariant) polarimetric properties. The conditions on source parameters ensuring that the sources of this class have uniform polarization in the source volume were also specified. In addition, particular sets of source parameters have been found for the cases of completely polarized and completely unpolarized sources. In the general case of non-uniform polarization several numerical examples were considered to illustrate the dependence of polarization states of a typical 3D EM GSM source on radial distance from its center.

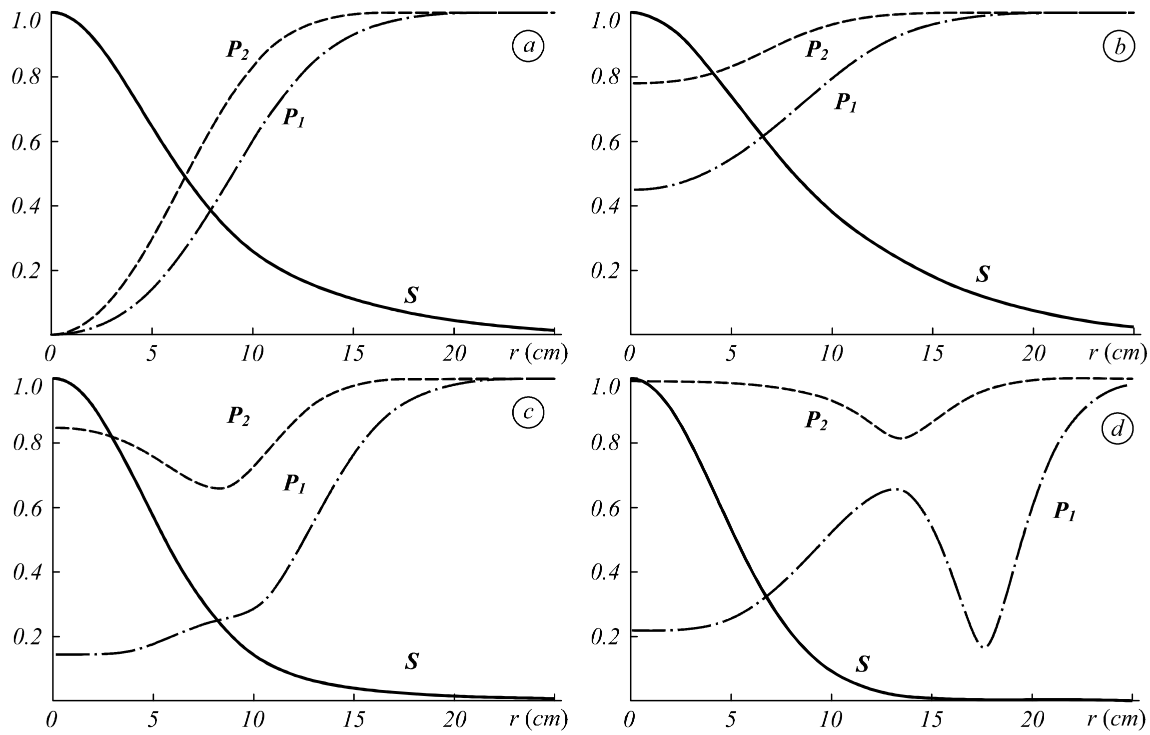


*Fig. 2. Spectral density S given in Eq. (6) and the two indices of polarimetric purity, $P_1$ and $P_2$, given by Eqs. (4) and (5), respectively, varying with radial distance r. (A) $A_x = A_y = A_z = 1$; $b_{xy} = b_{xz} = b_{yz} = 0$; (B) $A_x = 1$, $A_y = 1.5$, $A_z = 2$; (C) $A_x = 1$, $A_y = 1$, $A_z = 0.5$; (D) $A_x = 1$, $A_y = 1$, $A_z = 0.1$ in cases (B)-(D) the coefficients $b_{xy}$, $b_{xz}$ and $b_{yz}$ are given in (19)*

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