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QUANTIZATION IN STANDARD MODE-SELECTING ELEMENTS OF COMPUTER-SYNTHESIZED OPTICS

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Abstract—A theoretical investigation is carried out into the effect of quantization of complex transmittance on the performance of optical elements that form the transverse-mode structure of coherent illumination. A quantization model is constructed for the synthesis of optical elements by the methods of computer optics technology using various methods of coding with a carrier. Estimates are derived for the power efficiency of the optical elements subjected to quantization. To maintain accuracy of transverse-mode structure formulation, performance criteria are introduced and related to the parameters of quantization and the physical parameters of the light beam. An algorithm is developed to correct the perturbations due to quantization in the synthesis of mode-affecting elements on a computer. For the Gauss–Hermite modes, analytical expressions are derived and numerical estimates given for the power efficiency and performance in the formulation of a transverse-mode structure.

In computer-designed optics, the properties of mode affecting elements are governed to a certain extent by the quantization of their transmittance or reflection function [1–3]. Estimates of quantization errors in analysis of the transverse mode structure of coherent radiation were first reported by Golub *et al.* [1, 2].

This paper presents a theoretical investigation into the structure of light beams emerging from discrete mode-sensitive elements. It gives estimates of power efficiency in quantization and of accuracy in providing a desired transverse-mode structure.

TRANSFORMATION OF LIGHT BEAMS UNDER QUANTIZATION OF TRANSMITTANCE

The desired standard light beam with complex amplitude

$$(\mathbf{x}) = \sum_{p,l \in I_L} \xi_{pl} \psi_{pl}(\mathbf{x}), \quad \mathbf{x} \in G, \tag{1}$$

$$\xi_{pl} = \sqrt{\mu_{pl}} \exp(ib_{pl}), \tag{2}$$

containing $L \geqslant 1$ transverse modes $\psi_{pl}(\mathbf{x})$ with transverse indices $(p, l) \in I_L$, and having powers μ_{pl} and phases b_{pl}, may be obtained by passing an illuminating beam $E(\mathbf{x})$ through a complex spatial filter with the transmittance

$$W(\mathbf{x}) = \xi(\mathbf{x})/E(\mathbf{x}), \tag{3}$$

where I_L is the set of L double indices (p, l), and $\mathbf{x} = (x, y)$ are Cartesian coordinates in a section plane G of the beam.

The complex transmittance function

$$\Gamma(\mathbf{x}) = f\left(\frac{W(\mathbf{x})}{W_{\max}}, \mathbf{v}\right), \tag{4}$$

$$W_{\max} = \max_{\mathbf{x} \in G} |W(\mathbf{x})|, \tag{5}$$

treated in computer optics, satisfies the condition $|\Gamma(\mathbf{x})| \leqslant 1$. It is obtained by coding in some way f the function (3). The method of coding is selected such that in the first-order diffraction at an angle corresponding to the spatial frequency $\mathbf{v}$ the field $\gamma^{(1)}$

$$\gamma^{(1)}(\mathbf{x}) = c\xi(\mathbf{x}) \exp(i2\pi \mathbf{v}\mathbf{x}), \quad c = \text{const}. \tag{6}$$

is reconstructed proportional to $\Gamma(\mathbf{x})$. For this component $\Gamma^{(1)}$, the transmittance $\Gamma(\mathbf{x})$ corresponding to the first order diffraction must satisfy the relation

$$\Gamma^{(1)}(\mathbf{x}) = a\,\frac{W(\mathbf{x})}{W_{\max}}\exp(i2\pi\nu\mathbf{x}), \tag{7}$$

where

$$a = cW_{\max}. \tag{8}$$

At $\nu = 0$ the first order of diffraction becomes the zero order. Notice that the coefficient a is defined by the method of coding f only. It can be demonstrated, for example, that the amplitude and phase elements of mode sensitive optics with a carrier may be described by the same formulae as amplitude and phase diffraction gratings, respectively,

$$a = \beta\frac{\Delta A}{4}, \tag{9}$$

$$a = I_1\left(\frac{\psi_{\max}}{2}\beta\right), \tag{10}$$

where $\Delta E \in [0, 1]$ and $\varphi_{\max} \in [0, \pi]$ are the amplitude passband and phase shift of the recording medium, usually $\varphi_{\max} = \pi$, and $\beta \in [0, 1]$ is the modulation depth.

The transmittance function $\Gamma(\mathbf{x})$ is implemented by means of a discrete photomask. We assume that the mask is prepared by a photoplotter with discrete positioning and two-dimensional sweep. It provides $N_1 \times N_2$ counts with uniformly exposed pixels of size $\delta \times \delta$ each. The domain G is therefore a $d_1 \times d_2$ rectangle with $d_1 = N_1\delta$ and $d_2 = N_2\delta$.

The respective discrete transmittance function is piecewise linear and described by the relation [4]

$$\tilde{\Gamma}(\mathbf{x}) = \sum_{n=1}^{N_1}\sum_{m=1}^{N_2}\Gamma(\mathbf{x}_{nm})\chi_{nm}(\mathbf{x}), \tag{11}$$

where $\mathbf{x}_{nm} = (x_n, y_m)$ is the centre of the pixel G_{nm}, and

$$\begin{aligned}\chi_{nm}(\mathbf{x}) &= 1, \quad \text{for} \quad \mathbf{x} \in G_{nm},\\ &= 0, \quad \text{for} \quad \mathbf{x} \notin G_{nm}.\end{aligned} \tag{12}$$

From (7) and (11) it follows

$$\Gamma^{(1)}(\mathbf{x}) = a\,\frac{\tilde{W}(\mathbf{x})}{W_{\max}}\exp(i2\pi\nu\mathbf{x}), \tag{13}$$

where

$$\tilde{W}(\mathbf{x}) = \sum_{n=1}^{N_1}\sum_{m=1}^{N_2} W(\mathbf{x}_{nm})\chi_{nm}(\mathbf{x})\exp -i2\pi\nu(\mathbf{x}-\mathbf{x}_{nm}). \tag{14}$$

By virtue of (7) and (13) the function $\tilde{W}(\mathbf{x})$ approximates the complex transmittance function $W(\mathbf{x})$ given by (3).

The field in the first order formed by a mode sensitive element may be represented in the form

$$\tilde{\gamma}^{(1)}(\mathbf{x}) = E(\mathbf{x})\Gamma^{(1)}(\mathbf{x}) = c\eta(\mathbf{x})\exp(i2\pi\nu\mathbf{x}). \tag{15}$$

The function

$$\eta(\mathbf{x}) = \sum_{p,l\in I_L}\xi_{pl}\psi_{pl}(\mathbf{x}) \tag{16}$$

can be represented with the aid of (1) and (14) as

$$\eta(\mathbf{x}) = \sum_{p,l\in I_L}\xi_{pl}\varphi_{pl}(\mathbf{x}). \tag{17}$$

This function differs from the standard beam $\xi(\mathbf{x})$ in that the orthonormal mode functions $\psi_{pl}(\mathbf{x})$,

$(p, l) \in I_L$ are replaced by the perturbed mode functions

$$\varphi_{pl}(\mathbf{x}) = E(\mathbf{x}) \sum_{n=1}^{N_1} \sum_{m=1}^{N_1} \frac{\psi_{pl}(\mathbf{x}_{nm})}{E(\mathbf{x}_{nm})} \chi_{nm}(\mathbf{x}) \exp[-i2\pi\nu(\mathbf{x} - \mathbf{x}_{nm})]; \quad (p, l) \in I_L. \tag{18}$$

PERTURBATION OF MODE FUNCTIONS UNDER DISCRETIZATION

We shall interpret [5] the functions $\varphi_{pl}(\mathbf{x})$, $(p, l) \in I_L$, as the result of the perturbation

$$h_{pl}(\mathbf{x}) = \varphi_{pl}(\mathbf{x}) - \psi_{pl}(\mathbf{x}), \quad p, l \in I_L \tag{19}$$

acting on the orthonormal mode functions $\psi_{pl}(\mathbf{x})$, $(p, l) \in I_L$. The perturbations are caused by the discretization procedure and depend on the method used in coding the mode elements of computer-synthesized optics.

To investigate the mode structure of the beam subjected to discretization we introduce the matrix elements

$$H_{plp'l'} = \int_G h_{pl}^*(\mathbf{x}) \psi_{p'l'}(\mathbf{x}) \, d^2\mathbf{x}, \tag{20}$$

where the asterisk denotes the complex conjugate.

The perturbed mode functions are represented via the orthonormal functions by the formula

$$\varphi_{pl}(\mathbf{x}) = \psi_{pl}(\mathbf{x}) + \sum_{p',l'} H_{plp'l'}^* \psi_{p'l'}(\mathbf{x}), \tag{21}$$

where the sum is taken over all $p' = 0, 1, 2, \ldots$, and $l' = 0, 1, 2, \ldots$. Accordingly, the field $\eta(\mathbf{x})$ given by (17) may be represented in the form

$$\eta(\mathbf{x}) = \sum_{p,l} \eta_{pl} \psi_{pl}(\mathbf{x}), \tag{22}$$

where

$$\begin{aligned} \eta_{pl} &= \xi_{pl} + \sum_{(p',l') \in I_L} H_{p'l'pl}^* \xi_{p'l'} \qquad \text{for} \quad (p, l) \in I_L, \\ &= \sum_{(p',l') \in I_L} H_{p'l'pl}^* \xi_{p'l'}, \qquad \text{for} \quad (p, l) \notin I_L. \end{aligned} \tag{23}$$

The modes $\psi_{pl}(\mathbf{x})$, $(p, l) \in I_L$, necessary to construct the standard $\xi(\mathbf{x})$ given by Eq. (1), constitute merely the component

$$\eta_L(\mathbf{x}) = \hat{P}_L \eta(\mathbf{x}) = \sum_{(p,l) \in I_L} \eta_{pl} \psi_{pl}(\mathbf{x}) \tag{24}$$

of the field $\eta(\mathbf{x})$. Here $\hat{P}_L$ is the projecting operator on to the base functions ψ_{pl}, $(p, l) \in I_L$. For convenience we introduce L-dimensional vectors

$$\Xi_L = (\xi_{pl} : (p, l) \in I_L) \quad \text{and} \quad H_L = (\eta_{pl} : p, l \in I_L),$$

and the $L \times L$ matrices

$$\begin{aligned} E_L &= [\delta_{pp'} \delta_{ll'} : (p, l) \in I_L; (p', l') \in I_L], \\ H_L &= [H_{plp'l'} : (p, l) \in I_L; (p', l') \in I_L], \\ \Phi_L &= E_L + H_L, \end{aligned} \tag{25}$$

where $\delta_{pp'}$ is the Kronecker delta.

In vector notation the formula (23) becomes

$$H_L = \Phi_L^* \Xi_L = (E_L + H_L^*) \Xi_L, \tag{26}$$

where the asterisk denotes the Hermitian conjugate. In what follows we denote the scalar product as $(\cdot, \cdot)$ and use the symbol $|\cdot|$ for vector norms.

POWER EFFICIENCY UNDER DISCRETIZATION

If a mode affecting optical element is illuminated by the incident light

$$\varepsilon_i = \int_G |E(\mathbf{x})|^2 \, d^2\mathbf{x} \tag{27}$$

then the beam $\tilde{\gamma}^{(1)}(\mathbf{x})$ formed in the first order of diffraction contains both the desired modes

$$\gamma_L^{(1)}(\mathbf{x}) = \hat{P}_L \tilde{\gamma}^{(1)}(\mathbf{x}) = c\eta_L(\mathbf{x}) \exp(i2\pi\nu\mathbf{x}) \tag{28}$$

and undesirable modes of other orders. The intensity of the desired beam, ε_{1d}, i.e., used for generating the required modes ψ_{pl}, with $(p, l) \in I_L$, is defined as

$$\varepsilon_{1d} = \int_G |\gamma_L^{(1)}(\mathbf{x})|^2 \, d^2\mathbf{x} = c^2 \int_G |\eta_L(\mathbf{x})|^2 \, d^2\mathbf{x}$$

$$= c^2(H_L, H_L) = c^2(E_L - R_L(\Xi_L, \Xi_L)), \tag{29}$$

where

$$-R_L = \Phi_L \Phi_L^* - E_L = H_L + H_L^* + H_L H_L^*. \tag{30}$$

The efficiency in terms of the incident beam intensity (29) will be estimated as

$$\frac{\varepsilon_{1d}}{\varepsilon_i} = \frac{c^2}{\varepsilon_i} (E_L - R_L(\Xi_L, \Xi_L)). \tag{31}$$

For $L = 1$ when the standard beam contains only one transverse mode $\xi_{pl}\psi_{pl}(\mathbf{x})$, Eq. (31) takes the form

$$\frac{\varepsilon_{1d}}{\varepsilon_i} = \frac{|c_{pl}|^2 |\xi_{pl}|^2}{\varepsilon_i} |1 + H_{plpl}|^2. \tag{32}$$

where the double index "pl" relates to the quantities of the single-mode beam. Conversely, when the class of all standard beams $\xi(1)$ containing exactly L modes is considered, one may derive an estimate with the aid of quadratic forms [6]

$$\frac{c^2}{\varepsilon_i} [1 - \lambda_{\max}(R_L)] |\Xi_L|^2 \leqslant \frac{\varepsilon_{1d}}{\varepsilon_i} \leqslant \frac{c^2}{\varepsilon_i} [1 - \lambda_{\min}(R_L)] |\Xi_L|^2, \tag{33}$$

where $\lambda_{\min}(R_L)$ and $\lambda_{\max}(R_L)$ are the least and largest eigenvalues of the self-adjoint matrix R_L. Note that the bounds of this estimate are achieved in varying Ξ_L.

In the absence of discretization $H_L = R_L = 0$ and Eq. (32) reduces to

$$\frac{\varepsilon_{1pl}}{\varepsilon_i} = \frac{c^2}{\varepsilon_i} |\xi_{pl}|^2, \tag{34}$$

and the inequality (33) becomes

$$\frac{\varepsilon_{1pl}}{\varepsilon_i} = \frac{c^2}{\varepsilon_i} |\Xi_L|^2. \tag{35}$$

Thus, discretization of mode-affecting optical elements reduces the energy efficiency $[1 - \lambda_{\min}(R_L)]^{-1}$ to $[1 - \lambda_{\max}(R_L)]^{-1}$ fold owing to the diffraction scattering of the light beam into higher diffraction orders, including the $(1 + 2\mathrm{Re}H_{plpl} + |H_{plpl}|^2)^{-1}$ fold reduction at $L = 1$.

The higher order modes of the beam $\tilde{\gamma}^{(1)}$ carry away the intensity

$$\varepsilon_{1h} = \int_G |\gamma^{(1)}(\mathbf{x}) - \gamma_L^{(1)}(\mathbf{x})|^2 \, d^2\mathbf{x}$$

$$= \int_G |\gamma^{(1)}(\mathbf{x})|^2 \, d^2\mathbf{x} - \varepsilon_{1d}.$$

In view of (15), (17) and (29) we have

$$\varepsilon_{1h} = c^2 \int_G |\eta(\mathbf{x})|^2 \, d^2\mathbf{x} - \varepsilon_{1d}$$
$$= c^2((R_L - Q_L)\Xi_L \Xi_L), \tag{36}$$

where

$$Q_L = [Q_{plp'l'}: (p, l) \in I_L; p', l' \notin I_L],$$
$$Q_{plp'l'} = -\int_G \phi^*_{pl}(\mathbf{x}) \varphi_{p'l'}(\mathbf{x}) \, d^2\mathbf{x} + \delta_{pp'} \delta_{ll'}. \tag{37}$$

EFFECT OF DISCRETIZATION ON TRANSVERSE MODE STRUCTURE

A discrete mode-affecting optical element synthesized by the methods of computer optics technology alters the distribution of power among the generated modes $\psi_{pl}(\mathbf{x})$, $(p, l) \in I_L$, and the phases of the modes in the beam $\gamma_L^{(1)}(\mathbf{x})$ as compared with the required beam $\gamma^{(1)}(\mathbf{x})$ given by Eq. (6). Because discretization reduces the power efficiency, it would be wise to compare the mode structure of $\gamma_L^{(1)}(\mathbf{x})$ with that of $\theta\gamma^{(1)}(\mathbf{x})$, where $0 < \theta < 1$.

Imagine a virtual interferometer adjusted to the zero fringe for two interfering beams $\theta\gamma^{(1)}(\mathbf{x})$ and $-\gamma_L^{(1)}(\mathbf{x})$. The intensity fringe function in the interferogram is given by the formula

$$|\theta\gamma^{(1)}(\mathbf{x}) - \gamma_L^{(1)}(\mathbf{x})|^2,$$

and the light beam of the difference interferogram

$$\Delta^2 = \int_G |\theta\gamma^{(1)}(\mathbf{x}) - \gamma_L^{(1)}(\mathbf{x})|^2 \, d^2\mathbf{x} \tag{38}$$

may be viewed as a performance measure in synthesizing the mode structure.

Making use of (6), (24) and the property of orthonormality of the mode functions, Eq. (38) can be transformed to the form

$$\Delta^2 = c^2 |\theta \Xi_L - H_L|^2. \tag{39}$$

This criterion Δ^2 takes on the minimal value

$$\Delta^2_{\min} = c^2[(H_L, H_L) - \theta^2(\Xi_L, \Xi_L)]$$
$$= c^2([(1 - \theta^2)E_L R_L]\Xi_L, \Xi_L) \tag{40}$$

at the optimal value of

$$\theta = \frac{\operatorname{Re}(\Xi_L, H_L)}{(\Xi_L, \Xi_L)} = 1 + \frac{\left(\dfrac{H_L + H_L^*}{2} \Xi_L, \Xi_L\right)}{(\Xi_L, \Xi_L)}. \tag{41}$$

Estimating the quadratic forms (40) and (41) with self-adjoint matrices we arrive at the following bounds for Δ^2 and θ:

$$c^2[1 - \theta^2 - \lambda_{\max}(R_L)]|\Xi_L|^2 \leqslant \Delta^2 \leqslant c^2[1 - \theta^2 - \lambda_{\min}(R_L)]|\Xi_L|^2 \tag{42}$$

$$1 + \lambda_{\max}\left(\frac{H_L + H_L^*}{2}\right) \leqslant \theta \leqslant 1 + \lambda_{\min}\left(-\frac{H_L + H_L^*}{2}\right). \tag{43}$$

The relative error of the transverse-mode structure formulation under discretization is

$$\delta^2 = \frac{\Delta^2}{\displaystyle\int_G |\theta\gamma^{(1)}(\mathbf{x})|^2 \, d^2\mathbf{x}} = \frac{((E_L - R_L)\Xi_L, \Xi_L)}{\theta^2(\Xi_L, \Xi_L)} - 1. \tag{44}$$

Another measure of accuracy is the mean square error in mode beam generation, *viz.*

$$\Delta_g^2 = \int_G |\gamma^{(1)}(\mathbf{x}) - \tilde{\gamma}^{(1)}(\mathbf{x})|^2 \, d^2\mathbf{x} \tag{45}$$

with the respective relative error

$$\delta_g^2 = \Delta_g^2 \Big/ \int_G |\gamma^{(1)}(\mathbf{x})|^2 \, d^2\mathbf{x}$$
$$= \int_G |\xi(\mathbf{x}) - \eta(\mathbf{x})|^2 \, d^2\mathbf{x} \Big/ \int_G |\xi(\mathbf{x})|^2 \, d^2\mathbf{x}. \tag{46}$$

Equations (17), (1), (37) and (19) used together yield the estimate

$$\lambda_{\min}(\tilde{Q}_L) \leqslant \delta_g^2 = \frac{(\tilde{Q}_L \Xi_L, \Xi_L)}{(\Xi_L, \Xi_L)} \leqslant \lambda_{\max}(\tilde{Q}_L), \tag{47}$$

where

$$\tilde{Q}_L = -(Q_L + H_L + H_L^*). \tag{48}$$

For a single-mode beam ψ_{pl} $(L = 1)$, the above estimates take the form

$$\theta_{pl} = 1 + \operatorname{Re} H_{plpl}, \tag{49}$$

$$\Delta_{pl}^2 = c_{pl}^2[1 - \theta^2 + 2 \operatorname{Re} H_{plpl} + |H_{plpl}|^2]|\xi_{pl}|^2$$
$$= |c_{pl}\xi_{pl}|^2(\operatorname{Im} H_{plpl})^2, \tag{50}$$

$$\delta_{pl}^2 = (\operatorname{Im} H_{plpl})^2/(1 + \operatorname{Re} H_{plpl})^2, \tag{51}$$

$$\delta_{pl}^2 = -(Q_{plpl} + 2 \operatorname{Re} H_{plpl}). \tag{52}$$

Note that for real values of H_{plpl} we have the equality

$$\frac{\varepsilon_{1d}}{\varepsilon_i} = \frac{c_{pl}|\xi_{pl}|^2}{\varepsilon_i} \theta_{pl}^2 \tag{53}$$

that indicates that the factor θ^2 directly characterizes the reduction of power efficiency due to discretization.

CORRECTION OF PERTURBATIONS DUE TO DISCRETIZATION IN STANDARD MODE OPTICAL ELEMENTS

The operation of discretization gives rise to a number of undesirable modes, and therefore the resultant mode structure differs from the desired one, the difference being given by criterion (44). While there is no way to eliminate the higher order modes the distribution of power and phases of the first modes with $L \geqslant 2$ may be improved. For this purpose, in the automatic design of mode optical elements on a computer, we shall use predistorted coefficients Ξ_L [5] instead of the standard coefficients $\tilde{\Xi}_L$ in Eq. (17). Then the coefficients H_L (26) of the modes ψ_{pl}, $(p, l) \in I_L$ in the representation (24) give way to the coefficients

$$\tilde{H}_L = \Phi_L^* \tilde{\Xi}_L, \tag{54}$$

characterizing the mode structure formed by elements of computer-synthesized optics.

If the pre-distortion is selected as [5]

$$\Xi_L = \Phi_L^{*-1} \Xi_L, \tag{55}$$

then

$$\tilde{H}_{pl} = \Xi_L,$$

that is, the mode structure with respect to the first L modes will be exactly as required, although the modes of higher orders are also present. Making use of the Neumann function for

$$\|H_L\| < 1 \tag{56}$$

helps to represent the operation (55) rather accurately as the series

$$\tilde{\Xi}_L^{(p)} = \Xi_L + \sum_{r=0}^{p} (-H_L^*)^2 \Xi_L$$

or as the recurrence relations [5]

$$\begin{aligned} \tilde{\Xi}_L^{(0)} &= \Xi_L, \\ \tilde{\Xi}_L^{(r)} &= (-H_L^*)\tilde{\Xi}_L^{(r-1)} + \Xi_L; \quad r = \overline{1, \ldots, p}. \end{aligned} \tag{57}$$

In performing the correction of order p, we get

$$\begin{aligned} \tilde{H}_L^{(p)} &= (E_L + H_L^*)\Xi_L^{(p)} \\ &= (E_L + H_L^*) \sum_{r=0}^{p} (-H_L^*)^r \Xi_L \\ &= [E_L + (-1)^p H_L^{p+1}]^* \Xi_L. \end{aligned} \tag{58}$$

This equation replaces Eq. (26). Accordingly, Eq. (44) for the δ^2 criterion takes on the form

$$\delta^{(p)^2} = ((E_L - R_L^{(p)})\Xi_L, \Xi_L)/[\theta^{(p)}]^2(\Xi_L, \Xi_L), \tag{59}$$

where

$$-R_L^{(p)} = (-1)^p[H_L^{p+1} + H_L^{*p+1}] + (H_L H_L^*)^{p+1}, \tag{60}$$

$$\theta^{(p)} = 1 + (-1)^p \frac{\left(\dfrac{H_L^{p+1} + H_L^{*p+1}}{2}\Xi_L, \Xi_L\right)}{(\Xi_L, \Xi_L)}. \tag{61}$$

It should be noted that at $p = 0$ Eqs (57)–(61) become respectively the formulae (41)–(44), (30).

DISCRETIZATION IN OPTICAL ELEMENTS MATCHED WITH GAUSS–HERMITE MODES

For the orthonormal Gauss–Hermite modes [7] with complex amplitude

$$\psi_{pl}(x, y) = \psi_p(x)\psi_l(y), \tag{62}$$

where

$$\psi_p(x) = E_{0p} H_p\left(\frac{\sqrt{2}x}{\delta}\right) \exp\left(-\frac{x^2}{\delta^2}\right), \tag{63}$$

$$E_{0p} = \psi_{max}/2^p p!, \qquad \psi_{max} = (1/\delta)\sqrt{2/\pi}, \tag{64}$$

the derivative can be represented by the expression

$$\psi_p'(x) = \frac{1}{\delta}[\sqrt{p}\psi_{p-1}(x) - \sqrt{p+1}\psi_{p+1}(x)], \tag{65}$$

which follows from the recurrence relations of the Hermite polynomials $H_p(\cdot)$ [8]. The relation (65) and the orthogonality of the normalized functions (62) enable us to derive simple expressions for the matrix elements of perturbations. For a plane incident wave ($E(\mathbf{x}) = 1$), the formulae (A10) and (A14) of the Appendix yield

$$\begin{aligned} Q_{plp'l'} = \frac{1}{12N_\delta^2} \{&\delta_{ll'}[\sqrt{pp'} + \sqrt{(p+1)(p'+1)})\delta_{pp'} - \sqrt{p(p'+1)}\delta_{p-2,p'} - \sqrt{(p+1)p'}\delta_{p+2,p'}] \\ &+ \delta_{pp'}[(\sqrt{ll'} + \sqrt{(l+1)(l'+1)}\delta_{ll'} - \sqrt{l(l'+1)}\delta_{l-2,l'} - \sqrt{(l+1)l'}\delta_{l+2,l'}]\}, \end{aligned} \tag{66}$$

$$H_{plp'l'} = -Q_{plp'l'} + [\mathrm{sinc}(1/N_v)\,\mathrm{sinc}(1/\tilde{N}_v) - 1]\delta_{pp'}\delta_{ll'} + \frac{i}{N_\sigma}\left\{\frac{\mathrm{sinc}(1/N_v) - \cos(\pi/N_v)}{2\pi/N_v}\,\mathrm{sinc}(1/N_v)\delta_{ll'}[\sqrt{p'}\delta_{p+1,p'} - \sqrt{p'+1}\delta_{p-1,p'}] + \sin(1/N_v)\frac{\mathrm{sinc}(1/N_v) - \cos(\pi/\tilde{N}_v)}{2\pi/N_v}\delta_{pp'}[\sqrt{l'}\delta_{l+1,l'} - \sqrt{l'+1}\delta_{l-1,l'}]\right\}, \quad (67)$$

where $\mathrm{sinc}(\xi) = \sin(\pi\xi)/\pi\xi$,

$$N_\sigma = \sigma/\delta, \quad (68)$$

$$N_v = 1/v_x\delta, \quad \text{and} \quad \tilde{N}_v = 1/v_y\delta. \quad (69)$$

The quantities N_σ, N_v and $\tilde{N}_v$ indicate how many resolution elements may be placed within the radius of the fundamental mode, and within the period of the carrier along the x axis and the y axis.

Consider an illustrative example. Suppose that an optical element of size $d_1 = d_2 = d$ is to be fabricated that has to form a single Gauss–Hermite mode with parameter σ from a plane wave of illumination. Assume that the phase coding method will be used with the carrier $v_x = v$, $v_y = 0$, and the resolution δ is specified.

Observing that

$$W_{\max}/\xi_{pl} \leqslant \psi_{\max} = (1/\sigma)\sqrt{2/\pi} \quad (70)$$

we resort to the formulae (49–52), (32) and (66–69) to obtain the design relations ($\beta = 1$) as follows:

$$\delta^2_{g,pl} = 2[1 - \mathrm{sinc}(1/N_v] + (p+l+1)/6N_\sigma^2 \simeq \pi^2/3N_v^2 + (p+l+1)/6N_\sigma^2, \quad (71)$$

$$\frac{\varepsilon_{1d,pl}}{\varepsilon_i} = \frac{\varepsilon_{1pl}}{\varepsilon_i}[\mathrm{sinc}(1/N_v) - (p+l+1)/6N_\sigma^2], \quad (72)$$

where

$$\frac{\varepsilon_{1pl}}{\varepsilon_i} = \frac{1}{2}I_1^2\left(\frac{\pi\beta}{2}\right)\left(\frac{2\sigma}{d}\right)^2 \quad (73)$$

is the energy efficiency of the mode element in the absence of quantization.

For $\delta \to 0$ we have $N_v \to \infty$ and $N_\sigma \to 0$. For a specific $\delta > 0$, the leading terms in (71) and (72) describe the quantization error for the carrier frequency, while the second terms give the error of sampling of the mode function ψ_{pl}. Tables 1 and 2 summarize the values for the performance criteria (71) and (72).

As the order $(p+1)$ of mode ψ_{pl} increases, the power efficiency $\varepsilon_{1d,pl}/\varepsilon_i$ decreases and the mean square deviation δ_g^2 increases. Given an admissible drop of the power efficiency

$$\chi = |\varepsilon_{1d,pl} - \varepsilon_{1pl}|/\varepsilon_{1pl}$$

and the maximum value of the root mean square deviation $\delta_{g,\max}$ we arrive at an estimate of the maximum order of the mode, which may be recorded onto the fabricated optical element, namely,

$$(p+1)_{\max} = \min(p_1, p_2), \quad (74)$$

Table 1. Energy efficiency reduction due to quantization $\varepsilon_{1d,pl}/\varepsilon_{1pl}$ ($N_v = 4$)

	$p+1$				
N_σ	0	5	10	50	100
5	0.797	0.740	0.684	0.318	0.053
10	0.807	0.792	0.778	0.664	0.536
20	0.810	0.806	0.804	0.785	0.762
30	0.810	0.808	0.807	0.794	0.776

Table 2. Dependence of mode element characteristics on carrier spatial frequency ($N_\sigma = 10$, $p + l = 10$)

N_ν	2	4	6	8	10	20
$\frac{\varepsilon_{1d,pl}}{\varepsilon_{1pl}}$	0.374	0.777	0.868	0.913	0.924	0.954
$\delta^2_{g,pl}$	0.758	0.218	0.118	0.070	0.058	0.028

where

$$p_1 = 6N_\sigma^2[\operatorname{sinc}(1/N_\nu) - \sqrt{1-\chi}] - 1, \tag{75}$$

$$p_2 = 6N_\sigma^2[\delta^2_{g,\max} - 2(1 - \operatorname{sinc}(1/N_\nu)] - 1. \tag{76}$$

The confinement of the mode width to about $\delta\sqrt{p+0.5} \times \sigma\sqrt{l+0.5}$ due to the dimensions of the optical element gives rise to the estimate

$$p \leqslant p_3, \qquad l \leqslant p_3, \qquad p_3 = 4(d/2\sigma)^2 - 1/2. \tag{77}$$

Letting $\chi = 0.2$, $N_\nu = \delta_g^2 = 0.2$, $d = 5$ μm, and $\delta = 25$ μm we obtain $p + l \leqslant 21$ at $N_\sigma = 26$, and $p + l \leqslant 2$ at $N = 10$.

At greater spatial frequencies of the carrier ν, i.e. at lower N_ν, the power efficiency $\varepsilon_{1d,pl}$ given by Eq. (72) decreases, while the mean square criterion increases in value, which implies an enhancement of quality of the complex amplitude distribution formed, and an increase in the proportion of the respective intensity.

Thus, quantization dictates that the spatial frequency ν should be minimized. In doing so one should not overlook, however, that the lower bound of ν is controlled by the conditions of separability of the zeroth and first orders.

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APPENDIX

ESTIMATION OF MATRIX ELEMENTS OF PERTURBATION

We estimate the matrix elements $H_{plp'l'}$ and $Q_{plp'l'}$ given by (20) and (37), respectively, for a plane wave of illumination of $E(\mathbf{x}) = \text{const}$. For $\nu = 0$, an estimate of $H_{plp'l'}$ has been furnished by Golub and Soifer [4]. We shall generalize on the method devised in this paper.

The perturbations (19) and (18) due to quantization are piecewise functions having various parameters in different resolution cells G_{nm}. Recognizing that the pixel is small we may expand $\psi_{pl}(\mathbf{x})$ into a Taylor series at $\mathbf{x}_{nm}$ and retain the linear terms, *viz.*

$$\psi_{pl}(\mathbf{x}) = \psi_{pl}(\mathbf{x}_{nm}) + (\mathbf{x} - \mathbf{x}_{nm})\nabla\psi_{pl}(\mathbf{x}_{nm}), \quad x \in G_{nm}. \tag{A1}$$

Substituting (A1) into (18) and (19) yields

$$h_{pl}(\mathbf{x}) = \psi_{pl}(\mathbf{x}_{nm})\{\exp[-i2\pi\nu(\mathbf{x} - \mathbf{x}_{nm})] - 1\} - (\mathbf{x} - \mathbf{x}_{nm})\nabla\psi_{pl}\mathbf{x}_{nm}, \quad \mathbf{x} \in G_{nm}. \tag{A2}$$

Now we come to estimating the matrix elements (20),

$$H_{plp'l'} = \int_G h^*_{pl}(\mathbf{x})\psi_{p'l'}(\mathbf{x})\,\mathrm{d}^2\mathbf{x}$$
$$= \sum_{n,m} \int_{G_{nm}} h^*_{pl}(\mathbf{x})\psi_{p'l'}(\mathbf{x})\,\mathrm{d}^2\mathbf{x}. \tag{A3}$$

Substituting (A1) and (A2) into (A3) and observing that

$$\int_{G_{nm}} (\mathbf{x}-\mathbf{x}_{nm})\,\mathrm{d}^2\mathbf{x} = 0, \tag{A4}$$

$$\int_{G_{nm}} (\mathbf{x}-\mathbf{x}_n)^2\,\mathrm{d}^2\mathbf{x} = \int_{G_{nm}} (y-y_m)^2\,\mathrm{d}^2\mathbf{x} = \delta^4/12, \tag{A5}$$

$$\int_{G_{nm}} \{\exp[+i2\pi\mathbf{v}(\mathbf{x}-\mathbf{x}_{nm})]-1\}\,\mathrm{d}^2\mathbf{x} = \delta^2[\mathrm{sinc}(v_x\delta)\,\mathrm{sinc}(v_y\delta)-1], \tag{A6}$$

$$\int_{G_{nm}} (\mathbf{x}-\mathbf{x}_{nm})\{\exp[+i2\pi\mathbf{v}(\mathbf{x}-\mathbf{x}_{nm})]-1\}\,\mathrm{d}^2\mathbf{x} = i\delta^2\mathbf{F}(v_x, v_y, \delta), \tag{A7}$$

where sinc (t) = $\sin(\pi t)/\pi t$,

$$\mathbf{F}(v_x, v_y, \delta) = \begin{pmatrix} F_0(v_x, v_y, \delta) \\ F_0(v_y, v_x, \delta) \end{pmatrix}, \tag{A8}$$

$$F_0(v_x, v_y, \delta) = \frac{\mathrm{sinc}(v_x\delta) - \cos(\pi v_x\delta)}{2\pi v_x}\,\mathrm{sinc}(v_y\delta), \tag{A9}$$

we get

$$H_{plp'l'} = \delta^2 \sum_{n,m} \psi^*_{pl}(\mathbf{x}_{nm})\psi_{p'l'}(\mathbf{x}_{nm})[\mathrm{sinc}(v_x\delta)\,\mathrm{sinc}(v_y\delta)-1] + i\psi^*_{pl}(\mathbf{x}_{nm})\nabla\psi_{p'l'}(\mathbf{x}_{nm})F(v_x, v_y, \delta) - \nabla\psi^*_{pl}(\mathbf{x}_{nm})\nabla\psi_{p'l'}(\mathbf{x}_{nm})\frac{\delta^2}{12}.$$

We approximate the integral sum by the integral and take into account the treatment of $\psi_{pl}(\mathbf{x})$ at (A1). Then

$$H_{plp'l'} = -\frac{\delta^2}{12}\int_G \nabla\psi^*_{pl}(\mathbf{x})\nabla\psi_{p'l'}(\mathbf{x})\,\mathrm{d}^2\mathbf{x} + [\mathrm{sinc}(v_x\delta)\,\mathrm{sinc}(v_y\delta)-1]\delta_{pp'}\delta_{ll'} + i\mathbf{F}(v_x, v_y, \delta)\int_G \psi^*_{pl}(\mathbf{x})\nabla\psi_{p'l'}(\mathbf{x})\,\mathrm{d}^2\mathbf{x}. \tag{A10}$$

To compute the matrix elements $Q_{plp'l'}$ given by (37) we use Eq. (18) at $E(\mathbf{x}) = \text{const}$. On integration we have

$$Q_{plp'l'} = \delta_{pp'}\delta_{ll'} - \delta^2 \sum_{n,m} \psi^*_{pl}(\mathbf{x}_{nm})\psi_{p'l'}(\mathbf{x}_{nm}). \tag{A11}$$

On the other hand, the condition of orthonormality of mode functions on recognizing (A1), (A4) and (A5) becomes

$$\begin{aligned}\delta_{pp'}\delta_{ll'} &= \int_G \psi^*_{pl}(\mathbf{x})\psi_{p'l'}(\mathbf{x})\,\mathrm{d}^2\mathbf{x} \\ &= \sum_{n,m}\int_G \psi^*_{pl}(\mathbf{x})\psi_{p'l'}(\mathbf{x})\,\mathrm{d}^2\mathbf{x} \\ &= \delta^2\sum_{n,m}\psi^*_{pl}(\mathbf{x}_{nm})\psi_{p'l'}(\mathbf{x}_{nm}) + \frac{\delta^4}{12}\sum_{n,m}\nabla\psi^*_{pl}(\mathbf{x}_{nm})\nabla\psi_{p'l'}(\mathbf{x}_{nm}).\end{aligned} \tag{A12}$$

Substituting (A12) in (A11) yields

$$Q_{plp'l'} = (\delta^4/12)\sum_{n,m}\nabla\psi^*_{pl}(\mathbf{x}_{nm})\nabla\psi_{p'l'}(\mathbf{x}_{nm}). \tag{A13}$$

Approximating this sum by the integral leads us finally to

$$Q_{plp'l'} = (\delta^2/12)\int_G \nabla\psi^*_{pl}(\mathbf{x})\nabla\psi_{p'l'}(\mathbf{x})\,\mathrm{d}^2\mathbf{x}. \tag{A14}$$

The formulae (A10) and (A14) provide estimates for the matrix elements of perturbations encountered in the main part of this study.