DESIGN OF LENSES FOR THE FOCUSING INTO A LINE

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Abstract

A method for designing a curvilinear lens for focusing into a line is proposed. The obtained solutions represent a complex refracting surface in an analytical form expressed through the eikonal distribution along the line. The calculation of a 3-D lens focusing into a line-segment is reduced to a simple design of cylindrical profile with required function of ray-correspondence.

1. Introduction

Reference [1] reports studies of complex reflective surface aimed at focusing into a line of desired shape. The design of reflecting surfaces is based on a special type of ray-correspondence between points on the line of focusing and those on the reflecting surface and uses the Fermat's principle for the surface reconstruction.

In the present paper we use similar approach for designing refracting surfaces. The presented method extends the known design procedures to the design of complex lens focusing a spherical beam into a line.

2. Structure of a light field with a caustic line

In this section we briefly consider the general properties of wavefront having a caustic in the form of a line L. Let the line L be given by the following parametric equation:

$$\mathbf{F}(\sigma) = \left(x(\sigma), y(\sigma), z(\sigma)\right). \quad (1)$$

It is well known that a circular cone of rays passes through each caustic curve point [1,2]. One half of the cone corresponds to the rays coming to the caustic line and the other - to the rays leaving it (Fig.1). The cone axis is tangent to the line L and the angle of the cone is determined by the derivative

$$\cos\left(\omega(\sigma)\right) = \frac{ds(\sigma)}{d\sigma} \quad (2)$$

of the eikonal distribution $s(\sigma)$ along the line L [1,2]. For the subsequent calculation it is convenient to employ the coordinate system related to the conical structure of the light field [1]. The system uniquely defines the position of point $\mathbf{M}$ of a ray through the parameters ψ, ρ and ρ (Fig.1). The ψ is the angle characterising the ray within the cone and counted from the plane Λ adjoining the caustic L. The parameter σ defines the angle of the cone and the ρ is the distance along the ray to the point M measured from the top of the cone.

The vector $\mathbf{M}(\rho,\sigma,\psi)$ is related to the vector $\mathbf{F}(\sigma)$ and to the unit ray vector $\mathbf{q}_0(\sigma,\psi)$ in the cone via the relationship

$$\mathbf{M}(\rho,\sigma,\psi) = \mathbf{F}(\sigma) - \mathbf{q}_0(\sigma,\psi)\rho\,. \quad (3)$$

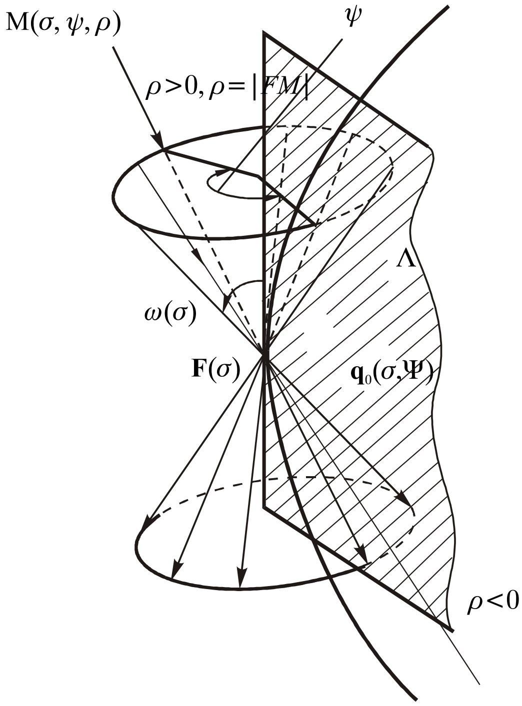

Fig.1. Conical structure of a light field near the caustic line

According to the above definitions of ρ and ψ we have

$$\mathbf{q}_0 = \mathbf{t}(\sigma)\cos\left(\omega(\sigma)\right) + \mathbf{n}(\sigma)\sin\left(\omega(\sigma)\right)\cos(\psi) + \mathbf{b}(\sigma)\sin\left(\omega(\sigma)\right)\sin(\psi), \quad (4)$$

where $\mathbf{t}(\sigma), \mathbf{n}(\sigma), \mathbf{b}(\sigma)$ are the basis vectors of the tangent, the normal and the binormal to the caustic curve L.

3. Calculation of optical elements focusing into a line

Let us first consider a calculation of a refractive surface S which transforms a spherical wave (focus $\mathbf{Q} = \left(q_x, q_y, q_z\right)$) spreading in a medium with refractive index n into a wavefront having in air (n=1) a pregiven caustic line L (see Fig. 2). The eikonal distribution s(σ) along the line L is supposed to be defined.

Since the refracting surface is two-dimensional and the line of focusing is one-dimensional, on the surface there is a 1-D line $P(\psi;\sigma)$ refracting the rays of the original spherical wave into a cone of rays coming to the same point $\mathbf{F}(\sigma)$ of the line L. The line $P(\psi;\sigma)$ is formed by the intersection of the cone (the cone angle is defined by the derivative $ds(\sigma)/d\sigma$) with the refractive surface of a lens, which transforms the spherical wave from **Q** into a convergent spherical beam with focus at the point $\mathbf{F}(\sigma)$ (Fig.2).

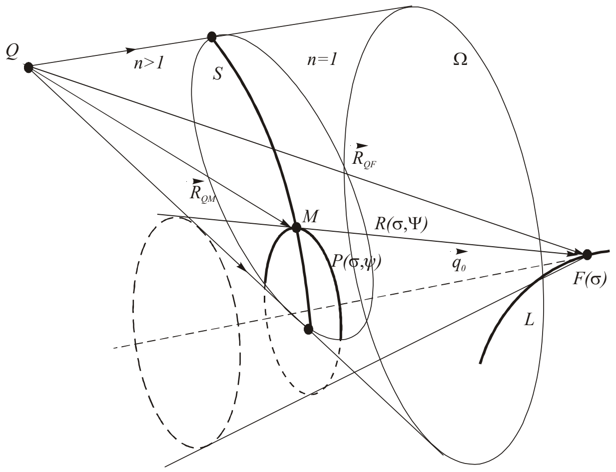

Fig.2. Geometry of focusing into a line

Obviously, it should consider only a portion of line $P(\psi;\sigma)$ located within the cone Ω defining an outline of the refractive surface. The family of lines $P(\psi;\sigma)$ with respect to the parameter σ forms the required surface S. Assume that **M** is the current point on the surface S. Then we can write the equation of the surface S in the following form:

$$\mathbf{R}_{QM}(\sigma,\psi) = \mathbf{R}_{QF}(\sigma) - \mathbf{q}_0(\sigma,\psi)R(\sigma,\psi)\,, \qquad (5)$$

where $\mathbf{R}_{QM}(\sigma,\psi)$ is the vector connecting the focus **Q** to the point **M** of the surface S,

$$\mathbf{R}_{QF}(\sigma) = \left(x(\sigma) - q_x,\, y(\sigma) - q_y,\, z(\sigma) - q_z\right)$$

is the vector connecting the focus **Q** to the point $\mathbf{F}(\sigma)$ of the caustic curve and $R(\sigma,\psi)$ is the distance from the point $\mathbf{F}(\sigma)$ along the ray $\mathbf{q}_0(\sigma,\psi)$ to the point **M** of the surface. To find the function $R(\sigma,\psi)$ in an explicit form, we apply for the line $P(\psi;\sigma)$ the consequence of Fermat principle which requires a constant optical path length from the original focus **Q** to the point $\mathbf{F}(\sigma)$. Using the condition of the constant optical path length we get

$$\pm n \cdot \mathbf{R}_{QM}(\sigma,\psi) + R(\sigma,\psi) = s(\sigma)\,, \qquad (6)$$

where the sign '+' corresponds to the divergent spherical wave with the real focus **Q** and sign '-' - to the convergent spherical beam with the imaginary focus **Q**. Substituting Eq.(6) into Eq.(7), we get a quadratic equation with respect to the distance $R(\sigma,\psi)$;

$$n^2\left(\mathbf{R}_{QF}(\sigma) - \mathbf{q}_0(\sigma,\psi)R(\sigma,\psi)\right)^2 = \\ = \left(s(\sigma) - R(\sigma,\psi)\right)^2 \qquad (7)$$

Resolving Eq. (7) we get the function $R(\sigma,\psi)$ in the form

$$R(\sigma,\psi) = \frac{1}{n^2-1}\Big(p(\sigma,\psi) \pm \\ \pm\sqrt{p^2(\sigma,\psi) + \left(n^2-1\right)\left(s^2(\sigma) - n^2\mathbf{R}_{QF}^2(\sigma)\right)}\Big)\,, \qquad (8)$$

where $p(\sigma,\psi) = n^2\left(\mathbf{R}_{QF}(\sigma), q_0(\sigma,\psi)\right) - s(\sigma)$ (9)

Thus, the refractive surface defined by Eqs. (5), (8), (9) provides focusing of the spherical wave into the line L.

Let us now consider a synthesis of the refractive surface S_p which focuses a plane wave of the direction $\mathbf{p} = \left(p_x, p_y, p_z\right), |\mathbf{p}| = 1$ into the line L. Although the plane beam is just a particular case of a spherical one, an equation of the surface S_p does not directly follow from Eqs. (5)-(9). Assume that $\mathbf{R}_M$ is the position vector of the current point **M** on the surface. Then we can write the equation of the surface S_p in the form:

$$\mathbf{R}_M(\sigma,\psi) = \mathbf{F}(\sigma) - \mathbf{q}_0(\sigma,\psi)R(\sigma,\psi) \qquad (10)$$

The condition of the constant optical path length

$$n \cdot \left(\mathbf{R}_M(\sigma,\psi), \mathbf{p}\right) = n \cdot \left(\mathbf{F}(\sigma) - q_0(\sigma,\psi), \mathbf{p}\right) = \\ = s(\sigma) - R(\sigma,\psi) \qquad (11)$$

allows one to obtain the function $R(\sigma,\psi)$ in the following form

$$R(\sigma,\psi) = \frac{s(\sigma) - n\cdot\left(\mathbf{F}(\sigma), \mathbf{p}\right)}{1 - n\cdot\left(\mathbf{q}_0(\sigma), \mathbf{p}\right)} \qquad (12)$$

The considered design of the refractive surface S provides focusing into a line of a spherical wave coming from the medium with refractive index *n*. Let us suppose that the incident spherical beam comes from air (*n*=1). Then the surface S is used as a second surface of an optical element aimed at focusing into the line L. At that time the first surface of the element must transform the incident spherical beam into a spherical beam inside the element. Thus, a calculation of the optical element focusing into a line consists in the calculation of a lens (the first surface) with subsequent implementation of Eqs. (5)-(9) to the design of the second surface.

Let us consider a calculation of the surface which transforms the incident spherical beam with focus **Q** into the spherical beam with focus $\mathbf{Q_1}$ spreading inside the medium with refractive index n (see Fig.3). Consider the vectors $\mathbf{r}, \mathbf{r}_1, \mathbf{l}$ connecting the points (Q,M), (Q_1,M) and (Q,Q_1), respectively. The condition of the constant optical path length allows one to write the surface equation in the following form

$$\pm n\cdot|\mathbf{r}| \pm |\mathbf{r}_1| = \pm n\cdot|\mathbf{r}_1 + \mathbf{l}| \pm |\mathbf{r}_1| = \\ = \pm n\cdot|\mathbf{r}| \pm |\mathbf{r} - \mathbf{l}| = s\,, \qquad (13)$$

where *s* is some value of eikonal equal to the optical path length from the focus **Q** to the focus $\mathbf{Q_1}$. The signs of the terms $n\cdot|\mathbf{r}|$ and $|\mathbf{r}_1|$ in Eq. (13) define two types of the beam (divergent or convergent) in accordance with the conventional rule of signs adopted in optics.

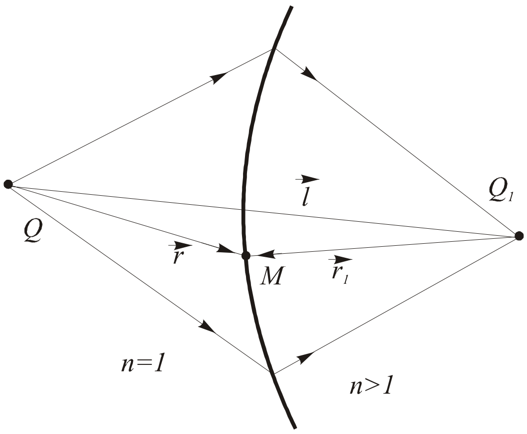

Fig.3. Calculation of lens surface with the focuses Q and Q_1

According to that rules the distances $|\mathbf{r}_1|$ and $|\mathbf{r}|$ are considered to be positive if the directions of $\mathbf{r}_1$ and $\mathbf{r}$ coincide with the directions of rays. On the contrary, the distances $|\mathbf{r}_1|$ and $|\mathbf{r}|$ are negative when the vectors $\mathbf{r}_1$ and $\mathbf{r}$ point against the spreading rays. The equations (13) give the following equations with respect to the $|\mathbf{r}_1|$ and $|\mathbf{r}|$

$$\pm n\cdot|\mathbf{r}_1+\mathbf{l}| = s\pm|\mathbf{r}_1|,$$
$$\pm|\mathbf{r}-\mathbf{l}| = s\pm n\cdot|\mathbf{r}| \tag{14}$$

The equations (14) can be easily reduced to conventional quadratic equations

$$\left(n^2-1\right)|\mathbf{r}_1|^2+2|\mathbf{r}_1|\left(n^2|\mathbf{l}|\cos(\theta_1)\pm s\right)+$$
$$+n^2|\mathbf{l}|^2-s^2=0, \tag{15}$$
$$\left(n^2-1\right)|\mathbf{r}|^2+2|\mathbf{r}|\left(|\mathbf{l}|\cos(\theta_1)\pm ns\right)+s^2-|\mathbf{l}|^2=0..$$

which give the lens surface with respect to the focus $\mathbf{Q}_1$

$$|\mathbf{r}_1(\theta_1)| = \frac{1}{\left(n^2-1\right)}\Big(\pm s-n^2|\mathbf{l}|\cos(\theta_1)\pm$$
$$\pm\sqrt{\left(n^2|\mathbf{l}|\cos(\theta_1)\pm s\right)^2+\left(n^2-1\right)\left(s^2-n^2|\mathbf{l}|^2\right)}\Big) \tag{16}$$

or with respect to the focus $\mathbf{Q}$

$$|\mathbf{r}(\theta)| = \frac{1}{\left(n^2-1\right)}\Big(\pm ns-|\mathbf{l}|\cos(\theta)\pm$$
$$\pm\sqrt{\left(\pm ns-|\mathbf{l}|\cos(\theta)\right)^2+\left(n^2-1\right)\left(|\mathbf{l}|^2-s^2\right)}\Big) \tag{17}$$

where θ_1 and θ are the angles between the vectors $\mathbf{r}_1,\mathbf{l}$ and the vectors $\mathbf{r},\mathbf{l}$, respectively.

The functions $|\mathbf{r}(\theta)|$ and $|\mathbf{r}_1(\theta_1)|$ depend on one variable only due to the radial symmetry of the lens with respect to the axis QQ_1.

4. Calculation of the lens focusing into a line-segment

Let us consider an example of designing a 3-D lens focusing into a line-segment $|x|\le d,\ y=0$ at $z=f$. To make a reasonable choice of the eikonal distribution $s(\sigma)$ along the line-segment let us first consider a design of a cylindrical profile focusing a spherical beam with focus $Q=(0,q_z)$ into the line-segment. Let $r(\sigma)$ be the radius-vector of the surface and let $x(\sigma)$ be a function describing a required ray correspondence between the angular coordinate $\sigma\in[\sigma_1,\sigma_2]$ of the incident ray and the coordinate x on the line-segment at $z=f$ (Fig. 4).

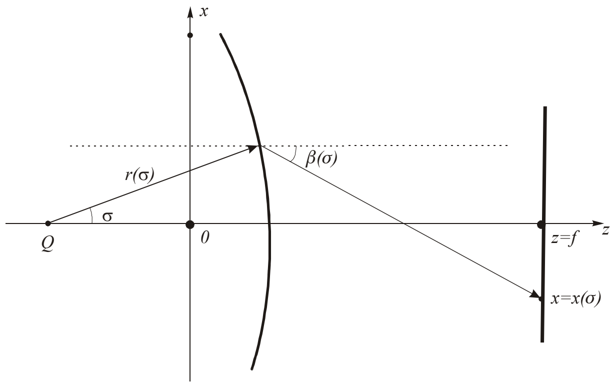

Fig.4. Geometry of focusing into a line-segment

Using the refraction law one can obtain for the function $r(\sigma)$ the differential equation of the first order resolved with respect to the derivative

$$\frac{dr(\sigma)}{d\sigma}=\pm r(\sigma)\frac{\sin(\sigma-\beta(\sigma))}{n-\cos((\sigma-\beta(\sigma)))},$$
$$\sigma\in[\sigma_1,\sigma_2],\ r(\sigma_1)=r_0 \tag{18}$$

where

$$\beta(\sigma)=\arctan\left(\frac{r(\sigma)\sin(\sigma)-x(\sigma)}{f-r(\sigma)\cos(\sigma)+q_z}\right) \tag{19}$$

The sign in Eq. (18) defines the diverging ('-') or convergent ('+') incident spherical beams. Eqs. (18) and (19) assumes existence of diverging and converging types of refracted surfaces. For the converging type surface $\beta(\sigma)>0$ and the reflected ray intersects the z-axis. For the surface of diverging type $\beta(\sigma)<0$ and the reflected ray does not intersect the optical axis. It is interesting to note that for n=-1 Eq. (18) reduces to well known equation for calculus of cylindrical reflectors [3].

The 2-D profile $r(\sigma)$ realising the required function of ray correspondence $x(\sigma)$can be considered as the XOZ section of the 3-D lens. This interpretation allows one to define the eikonal for the 3-D lens in the form

$$s(\sigma)=r(\sigma)+\frac{f-r(\sigma)\cos(\sigma)+q_z}{\cos(\beta(\sigma))}. \tag{20}$$

Since the angles of the cones are given by the function $\beta(\sigma)$, the direction of the coming ray in the cone corresponding to the point $x(\sigma)$ takes the form:

$$\mathbf{q}_0(\sigma,\psi) = \left(\sin(\beta(\sigma)),\ \cos(\beta(\sigma))\cdot\cos(\psi),\ \cos(\beta(\sigma))\cdot\sin(\psi)\right). \quad (21)$$

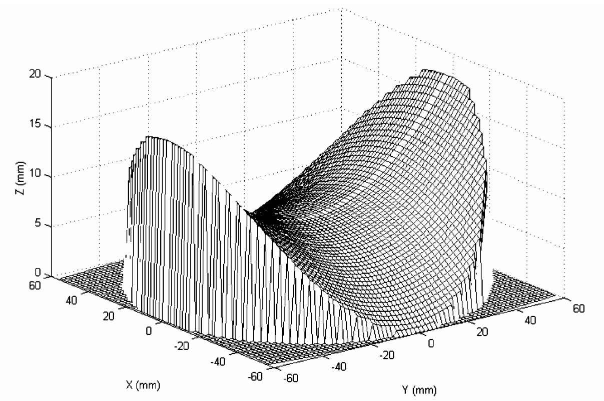

Fig.5. The lens (radius R=60mm) focusing convergent spherical beam (focus Q=(0,0,450mm)) at the plane z=12500mm into a line-segment of angular size α=35^0.

Fig. 5 depicts the calculated lens surface focusing the convergent spherical beam into the line-segment for the following parameters; radius of the surface R=60mm, distance to the plane of focusing f=12500mm, angular size of line-segment α=35^0, focus of illuminating beam **Q**=(0,0,450) mm. The surface was designed on the basis of the cylindrical profile focusing into a line-segment with the linear function of ray-correspondence $x(\sigma)$ (Fig. 6).

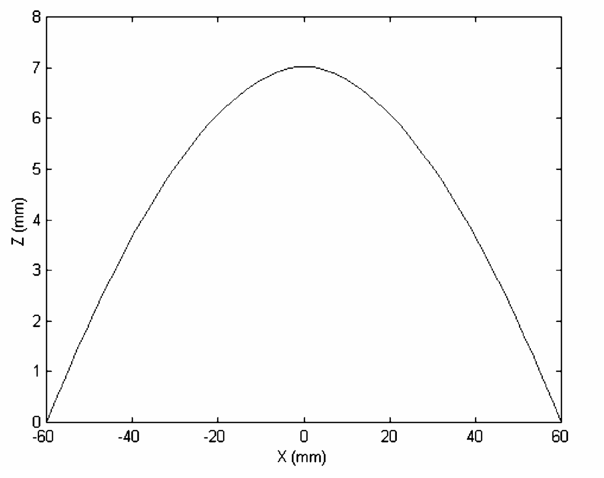

Fig.6. The XOZ section of 3-D lens shown on Fig.5.

Let us remind that the surface consists of the lines $P(\psi;\sigma)$. Each line is formed by the intersection of the cone with the lens surface, which focuses the spherical wave from the point **Q** into the point $(x(\sigma), 0, f)$ of the line-segment. Fig. 7 depicts the projections of the lines $P(\psi;\sigma)$ on the plane $z=0$. Following the terminology used in the theory of focusators, we call projections of the lines $P(\psi;\sigma)$ the layers [1,4]. Fig. 7 shows that the layers $L(\psi;\sigma)$ are close to straight lines perpendicular to the line-segment.

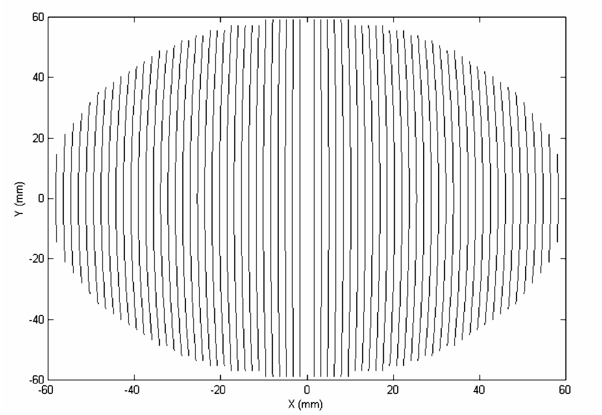

Fig.7. The systems of layers upon focusing into the line-segment

Let us emphasise that in spite of a strongly non-paraxial case (angular size of focal line is 35^0) the layers are close to the straight lines. As a result of our investigations, we have found that straight lines approximate the layers $L(\psi;\sigma)$ with a good accuracy upon focusing into a line with angular size $\alpha < 35 \div 40^0$ for the 'lens-focal plane' distance f > 5000mm. Thus, for the applications such as a design of automotive head-lamps the layers could be considered as the straight lines. This fact allows one to find the function $x(\sigma)$ in Eqs. (18), (19) from the condition of the formation of a desired linear density $I(\sigma)$ along the line-segment. To clarify the conception of linear density we shall consider two infinitely close points on a caustic line and a ray tube rested on these two points. Next, we shall find the ratio of the light flux contained in the given ray tube to the length of the line lying between the two points. The resulting quantity is the linear density of light field on the caustic line.

In what follows we assume the light flux contained between the two curves $P(\psi;\sigma)$ and $P(\psi;\sigma+\Delta\sigma)$ on the surface S to be equal to the light flux contained between the corresponding layers $L(\psi;\sigma)$ and $L(\psi;\sigma+\Delta\sigma)$. This assumption is quite true for a plane incident beam. In our opinion the above assumption has an acceptable accuracy practically for any real head-lamp devices including a simple parabolic, elliptical or hyperbolic reflector forming a spherical incident beam and a refractive lens creating a required line-segment. Within the above assumption we apply the light-flux conservation law and obtain for the determination of the $x(\sigma)$ function the following equation

$$\frac{dx(\sigma)}{d\sigma} = \frac{q_z}{I(x)\cdot\cos^2(\sigma)} \times \left(\int_{-\sqrt{R^2-(q_z\tan(\sigma))^2}}^{\sqrt{R^2-(q_z\tan(\sigma))^2}} E_0(q_z\tan(\sigma), y, 0)\,dy \right) \quad (22)$$

where $E_0(x, y, 0)$ is the illumination on the plane z=0 and R is the radius of the surface.

Thus, the design of the optical elements focusing into a line-segment consists of the following steps

1. Design of the first surface transforming an incident spherical wave into a spherical wave inside the element (Eqs. (16), (17)).

2. Calculation of the ray-correspondence function $x(\sigma)$ providing a required energy distribution along the line-segment (Eq. (22)).

3. Calculation of the cylindrical profile (Eqs.(18), (19)) corresponding to XOZ section of the 3-D profile and defining the eikonal distribution $s(\sigma)$ (Eq. (20)).

4. Design of 3-D surface of Eqs. (5), (8) and (9) on the basis of calculated eikonal $s(\sigma)$.

The considered problem of focusing into a line-segment can be treated as a reference problem of the generation of a complex line. The complex line may be approximated by a set of line-segments. To focus into the set of line segments, one can employ segmented lenses. In that case, the lens's aperture is broken down into segments (in accordance with the number of line-segments), each of them focusing into a corresponding part of the complex line under synthesis.

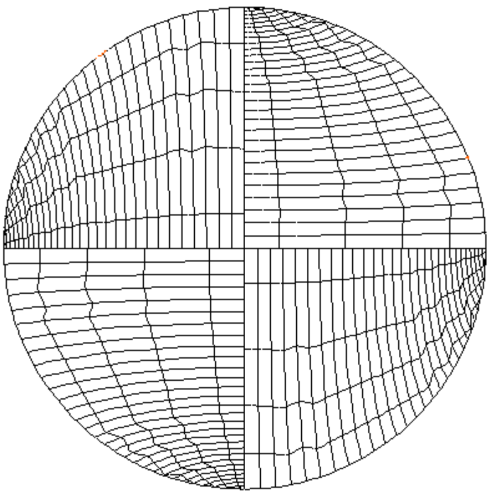

*Fig.8. The XOY view of the segmented lens (radius R=10mm) focusing divergent spherical beam(focus **Q**=(0,0,-18.7) mm) at the plane z=250mm into a cross of angular size α=30^0.*

Fig. 8 depicts the XOY view of the calculated segmented lens focusing the divergent spherical beam into the cross.

The lens consists of four angular segments, each segment focuses into the corresponding line-segment of the cross. The calculation has been done for the following parameters; radius of the surface R=10mm, distance to the plane of focusing f=250mm, angular size of line-segment α=30^0, focus of illuminating beam **Q**=(0.0 – 18.7) mm.

The illumination formed by the lens for the incident beam of "white LED" with gaussian intensity is shown on Fig. 9. The illumination was calculated within the ray-tracing software ASAP_BRO. The result of calculation shows the high quality of focusing and confirms the robustness of the developed method.

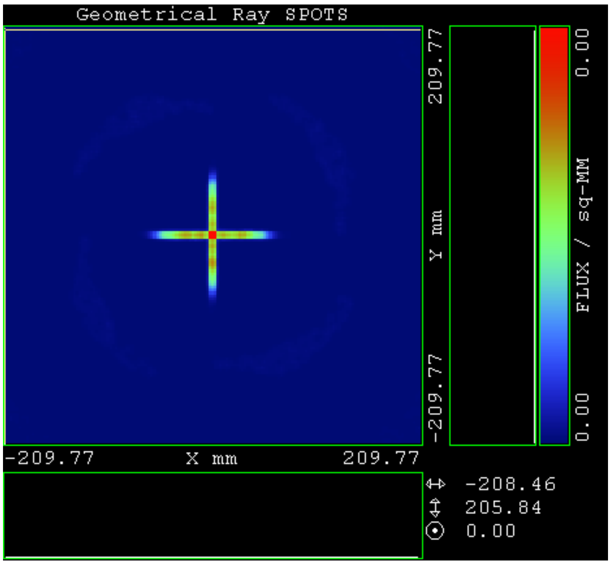

Fig.9. The calculated illumination created by the lens on Fig.8 for the "white LED" with gaussian intensity.

5. References

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